

Multiple-field inflation and the CMB

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Abstract. In this paper, we investigate some consequences of multiple-field inflation for the cosmic microwave background radiation (CMB). We derive expressions for the amplitudes, the spectral indices and the derivatives of the indices of the CMB power spectrum in the context of a very general multiple-field theory of slow-roll inflation, where the field metric can be non-trivial. Both scalar (adiabatic, isocurvature and mixing) and tensor perturbations are treated and the differences with single-field inflation are discussed. From these expressions, several relations are derived that can be used to determine the importance of multiple-field effects observationally from the CMB. We also study the evolution of the total entropy perturbation during radiation and matter domination and the influence of this on the isocurvature spectral quantities.

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1. Introduction

One of the greatest successes of the concept of inflation [1, 2, 3, 4, 5] is that it can give an explanation for the existence of small density fluctuations in an otherwise homogeneous universe. These small fluctuations, which are the gravitational seeds for the formation of large-scale structures, are observed in the cosmic microwave background radiation (CMB). Of course this works two ways: the observed amplitudes and slopes of the fluctuation spectra give us some observational constraints on the otherwise rather elusive parameters in inflation models, and thus on the parameters in the underlying high-energy theories. Hence it is important to have expressions for these spectral quantities in terms of the parameters of the inflation models. The most accurate observations of the CMB to date are those from the Wilkinson Microwave Anisotropy Probe (WMAP) mission [6] (see [7, 8, 9]).

This paper is a sequel to our paper [10]. In that paper a general theory of inflation, with an arbitrary number of scalar fields that take values on a possibly non-trivial field manifold, was considered (motivated by (string-inspired) supergravity theories where the Kähler potential leads to non-minimal kinetic terms). We treated both the background, paying special attention to the concept of slow roll, and the scalar perturbations. After deriving expressions valid to first order in slow roll for the perturbation quantities at the end of inflation, we also considered what happens after inflation. This finally led to

expressions for the correlators of the adiabatic and isocurvature scalar perturbations, as well as of the mixing between them, at the time of recombination (basically these are the scalar amplitudes of the CMB power spectrum). An example of a quadratic potential was included to illustrate the theory.

In this paper, these results are extended in basically two ways. In the first place, expressions for the spectral indices \tilde{n} and their derivatives $d\tilde{n}/d \ln k$ are given next to those for the amplitudes, and results for the tensor perturbations are added as well. Using these results we derive an expression that can be used to observationally test the importance of multiple-field effects in general, as well as various consistency relations for the one- and two-field cases. Secondly, the behaviour of the total entropy perturbation after inflation is studied to investigate under which conditions the usual assumption of a constant isocurvature perturbation is justified and what happens to the isocurvature spectral quantities if this is not the case.

A large number of references to other papers on inflationary perturbation theory was given in the introduction of [10]. More recently, a second-order treatment of the single-field case was given in [11], while the authors of [12, 13] used non-slow-roll techniques in the single-field case. Many aspects of the two-field case were recently discussed in [14, 15, 16, 17] (especially the last of these contains a very large number of references to earlier papers). An extensive quantum treatment, including backreaction, of $O(N)$ -symmetric multiple-field inflation models was given in [18]. Previous work on consistency relations for two-field inflation can be found in [19, 20, 21, 15, 16].

The outline of the paper is as follows. Section 2 deals with the general theory of multiple-field inflation. In three subsections the background, scalar perturbations and tensor perturbations are discussed. Most of this (except for the tensor perturbations) was derived in [10], so that this section is kept brief, only setting up the notation and giving the results that are used in the rest of the paper. Derivations are only given for those results that were not present in [10]. In section 3 (plus the appendix), expressions for the scalar (adiabatic, isocurvature and mixing) and tensor amplitudes, spectral indices, and the derivatives of the spectral indices are derived. The results are discussed with special attention to the differences with single-field inflation. In subsection 3.3, a number of important relations between these quantities are derived that are relevant for observationally testing the significance of multiple-field effects. Section 4 is devoted to the total entropy perturbation, which enters as a source term in the equation of motion for the gravitational potential after inflation in a system with multiple components originating from multiple fields during inflation, and thus determines the isocurvature perturbation. An equation for its time derivative is derived and the resulting time dependence is discussed. Finally, section 5 contains the conclusions. Some of the results in this paper have appeared in [22].

2. Multiple-field inflation

After defining the general setup of the theory, the results for all the relevant perturbation quantities at the time of recombination are given in this section. For the derivation of the scalar results as well as for more details about the background quantities the reader is referred to our previous paper [10]; the derivation of the tensor results is given here. Readers interested in seeing more intermediate steps in the calculations might also want to take a look at [22].

2.1. Background

We consider the situation where the matter content of the universe consists of an arbitrary number of real scalar fields ϕ^a , which are represented as a vector $\boldsymbol{\phi}$ of scalar field components. These fields are the coordinates on a possibly non-trivial field manifold with field metric \mathbf{G} . The field is separated into a classical homogeneous background part and a small quantum perturbation part: $\boldsymbol{\phi}^{\text{full}}(\eta, \mathbf{x}) = \boldsymbol{\phi}(\eta) + \boldsymbol{\delta\phi}(\eta, \mathbf{x})$. All equations are linearized with respect to the perturbations. The associated Lagrangean with potential V is

$$\mathcal{L} = \sqrt{-g} \left(-\frac{1}{2} \partial^\mu \boldsymbol{\phi} \cdot \partial_\mu \boldsymbol{\phi} - V(\boldsymbol{\phi}) \right) \equiv \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \boldsymbol{\phi}^T \mathbf{G} \partial_\nu \boldsymbol{\phi} - V(\boldsymbol{\phi}) \right), \quad (1)$$

where T denotes the transpose and $g_{\mu\nu}$ is the spacetime metric. Because of the non-trivial field metric we have to define covariant derivatives instead of normal ones: $\nabla_b A^a \equiv A^a_{,b} + \Gamma^a_{bc} A^c$ and $\mathcal{D}_\mu A^a \equiv \partial_\mu A^a + \Gamma^a_{bc} \partial_\mu \phi^b A^c$ for the field derivative and the spacetime derivative of a vector A^a in field space.

For the metric part of the universe we make the following definition:

$$g_{\mu\nu} = a^2 \begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} - a^2 \begin{pmatrix} 2\Phi & 0 \\ 0 & 2\Phi \delta_{ij} \end{pmatrix} + a^2 \begin{pmatrix} 0 & S_j \\ S_i & 0 \end{pmatrix} + a^2 \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}, \quad (2)$$

where we have applied the longitudinal and vector gauges and used the (ij) -component ($i \neq j$) of the Einstein equation to set $\Psi = \Phi$ in the scalar part (see e.g. [23, 24]). The first term represents the homogeneous background (a flat Robertson-Walker metric with scale factor $a(\eta)$), the second term the scalar perturbation (with $\Phi(\eta, \mathbf{x})$ the gravitational potential), the third term the vector perturbation (with $\mathbf{S}(\eta, \mathbf{x})$ a divergenceless vector), and the fourth term the tensor perturbation (with $h_{ij}(\eta, \mathbf{x})$ a symmetric transverse traceless tensor, i.e. $h_{ij} = h_{ji}$, $h^i_{,i} = 0$ and $h^i_i = 0$). As proved in [25, 24] scalar, vector and tensor perturbations decouple to linear order and can be considered separately. We make use of three different time coordinates: comoving time t , conformal time η and the number of e-folds N , defined as

$$dt = a d\eta, \quad dN = H dt = \mathcal{H} d\eta, \quad H \equiv \frac{\dot{a}}{a}, \quad \mathcal{H} \equiv \frac{a'}{a} = aH, \quad (3)$$

where H and \mathcal{H} are the Hubble parameters associated with t and η , respectively, and a dot (prime) denotes a derivative with respect to comoving (conformal) time.

For giving a physical interpretation of the various scalar field components, it is very useful to define a basis on the field manifold. (It is also an important ingredient of the calculations, especially quantization, as explained in [10], where we introduced this basis.) We define the first basis vector \mathbf{e}_1 as the direction of the field velocity: $\mathbf{e}_1 \equiv \dot{\boldsymbol{\phi}}/|\dot{\boldsymbol{\phi}}|$. Next, \mathbf{e}_2 is the direction of that part of the field acceleration $\mathcal{D}_t \dot{\boldsymbol{\phi}}$ that is perpendicular to \mathbf{e}_1 , and this is extended to higher-order derivatives to define the other basis vectors. Once this basis is defined we can use projection operators to split vectors into their physical components. The most important ones are $\mathbf{P}^\parallel \equiv \mathbf{e}_1 \mathbf{e}_1^T \mathbf{G}$ and $\mathbf{P}^\perp \equiv \mathbb{1} - \mathbf{P}^\parallel$ that make it possible to distinguish between effectively single-field (\mathbf{e}_1) and truly multiple-field effects. The basis vectors are not constant in time; their time derivatives are given by

$$\mathcal{D}_t \mathbf{e}_n = H \left(\frac{\tilde{\eta}_{n+1}^{(n+1)}}{\tilde{\eta}_n^{(n)}} \mathbf{e}_{n+1} - \frac{\tilde{\eta}_n^{(n)}}{\tilde{\eta}_{n-1}^{(n-1)}} \mathbf{e}_{n-1} \right), \quad (4)$$

where $\tilde{\eta}_n^{(n)} \equiv \mathbf{e}_n \cdot \tilde{\boldsymbol{\eta}}^{(n)}$, with $\tilde{\boldsymbol{\eta}}^{(n)}$ defined below.

The multiple-field generalization of the single-field slow-roll approximation was treated in [10] as well. We defined the following slow-roll functions:

$$\tilde{\epsilon} \equiv -\frac{\dot{H}}{H^2} = \frac{\frac{1}{2}\kappa^2 |\dot{\boldsymbol{\phi}}|^2}{H^2}, \quad \tilde{\boldsymbol{\eta}}^{(n)} \equiv \frac{\mathcal{D}_t^{n-1} \dot{\boldsymbol{\phi}}}{H^{n-1} |\dot{\boldsymbol{\phi}}|}, \quad (5)$$

and the short-hand notation $\tilde{\boldsymbol{\eta}} \equiv \tilde{\boldsymbol{\eta}}^{(2)}$ and $\tilde{\boldsymbol{\xi}} \equiv \tilde{\boldsymbol{\eta}}^{(3)}$. Here κ is the inverse reduced Planck mass: $\kappa^2 \equiv 8\pi G = 8\pi/M_P^2$. Using the basis discussed above we can take the components $\tilde{\eta}^\parallel \equiv \mathbf{e}_1 \cdot \tilde{\boldsymbol{\eta}}$ and $\tilde{\eta}^\perp \equiv \mathbf{e}_2 \cdot \tilde{\boldsymbol{\eta}}$ (by construction there are no other components), and similarly for $\tilde{\boldsymbol{\xi}}$, although there one has not just a single perpendicular direction, but two ($\tilde{\xi}_2$ and $\tilde{\xi}_3$). Using these definitions the equations of motion for the background quantities are:

$$H = \frac{\kappa}{\sqrt{3}} \sqrt{V} \left(1 - \frac{1}{3} \tilde{\epsilon} \right)^{-1/2}, \quad \phi_{,N} + \frac{1}{\kappa^2} \frac{\mathbf{G}^{-1} \nabla^T V}{V} = -\frac{\sqrt{2}}{3\kappa} \frac{\sqrt{\tilde{\epsilon}} (\tilde{\boldsymbol{\eta}} + \tilde{\epsilon} \mathbf{e}_1)}{1 - \frac{1}{3} \tilde{\epsilon}}, \quad (6)$$

where we have switched to using the number of e-folds N as time coordinate because it simplifies the equation. These equations are still exact; the slow-roll functions have only been used as short-hand notation. However, the next step is the slow-roll approximation: we assume that the slow-roll functions are small so that we can set up a series expansion in terms of them. ($\tilde{\epsilon}$ and $\tilde{\boldsymbol{\eta}}$ are assumed to be first-order quantities, while $\tilde{\boldsymbol{\eta}}^{(n)}$ is of order $n-1$.) Note that the time derivatives of the slow-roll functions are of one order higher in slow roll:

$$\begin{aligned} \tilde{\epsilon}_{,N} &= 2\tilde{\epsilon}(\tilde{\epsilon} + \tilde{\eta}^\parallel), & \mathcal{D}_N \tilde{\boldsymbol{\eta}}^{(n)} &= \tilde{\boldsymbol{\eta}}^{(n+1)} + ((n-1)\tilde{\epsilon} - \tilde{\eta}^\parallel) \tilde{\boldsymbol{\eta}}^{(n)}, \\ (\tilde{\eta}^\parallel)_{,N} &= \tilde{\xi}^\parallel + (\tilde{\eta}^\perp)^2 + (\tilde{\epsilon} - \tilde{\eta}^\parallel) \tilde{\eta}^\parallel, & (\tilde{\eta}^\perp)_{,N} &= \tilde{\xi}_2 + (\tilde{\epsilon} - 2\tilde{\eta}^\parallel) \tilde{\eta}^\perp. \end{aligned} \quad (7)$$

2.2. Scalar perturbations

The derivation of expressions for the scalar perturbations was the main topic of [10]. Because the equations of motion for the gravitational potential and the field

perturbations do not decouple in the multiple-field case, we cannot ignore the latter as one can in the single-field case. The scalar-field perturbations are redefined and quantized as

$$\mathbf{q} \equiv a \left(\delta\phi + \frac{\Phi}{\mathcal{H}} \phi' \right), \quad \hat{q} = Q\hat{a}^\dagger + Q^*\hat{a}, \quad (8)$$

with \hat{a}^\dagger and \hat{a} constant creation and annihilation operator vectors and $Q(\eta)$ a matrix function that satisfies a classical equation of motion. Here we switched to working in the explicit basis: non-bold versions of a vector or matrix represent those quantities in this basis, for example, $q^T = (q_1, q_2, \dots) = (\mathbf{e}_1 \cdot \mathbf{q}, \mathbf{e}_2 \cdot \mathbf{q}, \dots)$. This means that the non-bold basis vectors are simple constant vectors: $e_1^T = (1, 0, 0, \dots)$, $e_2^T = (0, 1, 0, \dots)$, etc. In the rest of this section we will denote expressions like the one for \hat{q} in (8) as $\hat{q} = Q\hat{a}^\dagger + \text{c.c.}$, with the c.c. meaning complex conjugate.

The results for the adiabatic and isocurvature parts (see below for a definition as well as for some remarks about assumptions made in the derivation) of the gravitational potential in terms of Fourier modes at the time of recombination, valid to first order in slow roll for modes with $k \ll \mathcal{H}_{\text{rec}}$ (super-horizon modes), are

$$\hat{\Phi}_{\mathbf{k} \text{ ad}}(t_{\text{rec}}) = \frac{3}{5} \frac{\kappa}{2k^{3/2}} \frac{H_{\mathcal{H}}}{\sqrt{\tilde{\epsilon}_{\mathcal{H}}}} (e_1^T + U_{Pe}^T) E_{\mathcal{H}} \hat{a}_{\mathbf{k}}^\dagger + \text{c.c.}, \quad (9)$$

$$\hat{\Phi}_{\mathbf{k} \text{ iso}}(t_{\text{rec}}) = \frac{1}{6} \frac{3}{5} \frac{\kappa}{2k^{3/2}} \frac{H_{\mathcal{H}}}{\sqrt{\tilde{\epsilon}_{\mathcal{H}}}} V_e^T E_{\mathcal{H}} \hat{a}_{\mathbf{k}}^\dagger + \text{c.c.} \quad (10)$$

Here we have defined $E_{\mathcal{H}} \equiv (1 - \tilde{\epsilon}_{\mathcal{H}})\mathbb{1} + (2 - \gamma - \ln 2)\delta_{\mathcal{H}}$, with γ the Euler constant, and[‡]

$$\delta \equiv \tilde{\epsilon} \mathbb{1} - \frac{\tilde{M}^2}{3H^2} + 2\tilde{\epsilon} e_1 e_1^T, \quad \tilde{\mathbf{M}}^2 \equiv \mathbf{G}^{-1} \nabla^T \nabla V - \mathbf{R}(\dot{\phi}, \dot{\phi}), \quad (11)$$

$$U_{Pe}^T \equiv 2\sqrt{\tilde{\epsilon}_{\mathcal{H}}} \int_{t_{\mathcal{H}}}^{t_e} dt' H \frac{\tilde{\eta}_e^\perp}{\sqrt{\tilde{\epsilon}}} \frac{a_{\mathcal{H}}}{a} e_2^T Q Q_{\mathcal{H}}^{-1}, \quad V_e^T \equiv \sqrt{\tilde{\epsilon}_{\mathcal{H}}} \frac{\sqrt{\tilde{\epsilon}_e} \tilde{\eta}_e^\perp}{\tilde{\epsilon}_e + \tilde{\eta}_e^\parallel} \frac{a_{\mathcal{H}}}{a_e} e_2^T Q_e Q_{\mathcal{H}}^{-1}. \quad (12)$$

The subscript \mathcal{H} means that the quantity has to be evaluated at the time $t_{\mathcal{H}}$ during inflation when $\mathcal{H} = k$ (often called the time of horizon crossing). The quantity $Q_{\mathcal{H}}$ is given by $Q_{\mathcal{H}} = E_{\mathcal{H}}/\sqrt{2k}$. Note that the slow-roll approximation was only used during a small transition region around the time $t_{\mathcal{H}}$. The time t_e is the end of inflation. The vectors U_{Pe} and V_e have no components in the e_1 direction. The matrix \mathbf{R} is the curvature tensor on the field manifold and $[\mathbf{R}(\dot{\phi}, \dot{\phi})]_a^{a} = R_{bcd}^a \dot{\phi}^b \dot{\phi}^c$.

The terms adiabatic and isocurvature relate to the evolution after inflation. During radiation and matter domination the equation of motion for super-horizon modes of Φ ,

$$\ddot{\Phi} + 4H(1 + \frac{3}{4}c_s^2)\dot{\Phi} + \kappa^2(\rho c_s^2 - p)\Phi = -2\kappa^2(\rho c_s^2 - p)\tilde{S}, \quad (13)$$

has an inhomogeneous source term proportional to the total entropy perturbation \tilde{S} :

$$\tilde{S} \equiv \frac{1}{4} \frac{\delta p - c_s^2 \delta \rho}{p - c_s^2 \rho}, \quad (14)$$

[‡] For normalization reasons, we have removed a factor $\frac{1}{2}$ from the definition of V_e as compared with our original definition in [10].

with p and ρ the total pressure and energy density and the sound velocity $c_s^2 \equiv \dot{p}/\dot{\rho}$. The adiabatic perturbation is the homogeneous solution for Φ of this equation of motion, while the isocurvature perturbation is the particular solution generated by the source term, with the initial conditions that it is zero and has zero derivative at the beginning of the radiation-dominated era. If \tilde{S} is constant on super-horizon scales, we have the simple solution $\Phi_{\text{iso}} = -\frac{1}{5}\tilde{S}$ during the matter era, which was used in the derivation of equation (10). To perform the matching of these solutions with the ones at the end of inflation (to determine the constants of integration) we assumed an immediate transition to a radiation-dominated universe at the end of inflation, ignoring (p)reheating. Especially for the isocurvature perturbations this is expected to be a crude approximation (since V_e depends crucially on what happens at the end of inflation, see (12)), which needs further improvement, but the treatment of the perturbations during a more realistic transition at the end of inflation as well as during an epoch of (p)reheating is still under investigation.[§]

The behaviour of \tilde{S} during radiation and matter domination is investigated in section 4, to see if the assumption of constancy on super-horizon scales is justified. However, let us discuss here what happens with (10) in the case that \tilde{S} is not constant, but given by $\tilde{S}(t) = f(t)\tilde{S}_e$. Here $f(t)$ encodes both the possible time dependence of \tilde{S} during radiation and matter domination and a possible correction factor to the simple matching condition described above, and \tilde{S}_e is the total entropy perturbation at the end of inflation, given by [10, 22]

$$\tilde{S}_e = \frac{\kappa}{2\sqrt{2}} \left[\frac{\sqrt{\tilde{\epsilon}}}{\tilde{\epsilon} + \tilde{\eta}^{\parallel}} \tilde{\eta}^{\perp} \frac{q_2}{a} \right]_e. \quad (15)$$

A particular solution for the inhomogeneous equation (13) is now given by $\Phi_{\text{part}}(t) = -2F(t)\tilde{S}_e$, where ($w \equiv p/\rho$)

$$F(t) \equiv \frac{H}{a} \int^t dt' \frac{1+w}{\frac{5}{6} + \frac{1}{2}w} \frac{d}{dt'} \left(\frac{a}{H} \right) \int^{t'} dt'' f(t'') \frac{d}{dt''} \left(\frac{\frac{5}{6} + \frac{1}{2}w}{1+w} \right). \quad (16)$$

Adding the homogeneous solution $\Phi_{\text{hom}} = C(H/a) + D(H/a) \int^t dt' (1+w)a$, with appropriate constants C and D to satisfy the initial conditions, we finally obtain the new result for $\Phi_{\text{iso}}(t_{\text{rec}})$. It turns out that we can represent it by exactly the same expression as in equation (10), if we change the definition of V_e to

$$V_e^T \equiv \left[10F(t_{\text{rec}}) - 3 \left(3F(t_*) + \frac{\dot{F}(t_*)}{H_*} \right) \right] \sqrt{\tilde{\epsilon}_{\mathcal{H}}} \frac{\sqrt{\tilde{\epsilon}_e} \tilde{\eta}_e^{\perp}}{\tilde{\epsilon}_e + \tilde{\eta}_e^{\parallel}} \frac{a_{\mathcal{H}}}{a_e} e_2^T Q_e Q_{\mathcal{H}}^{-1}, \quad (17)$$

with t_* the beginning of the radiation-dominated era. Note that V_e has now lost the physical interpretation of a quantity completely determined by inflation, but mathematically it is very convenient to work with an expression for $\Phi_{\text{iso}}(t_{\text{rec}})$ in which only V_e changes, as opposed to changing the form of (10) while keeping the definition of

[§] Some study on the effects of preheating on the scalar perturbations has been done for specific models, see [26, 27, 28, 29, 30, 31, 32] and references therein, but different authors do not yet agree regarding the conclusions.

V_e fixed. In the case that $f(t) = 1$ for all t , one finds $F(t) = 1$ for all t , and (17) reduces to (12), as it should.

2.3. Tensor perturbations

Having reviewed the scalar perturbation results from [10] in the previous subsection, we now turn to tensor perturbations. Scalar-field perturbations cannot, by definition, generate tensor perturbations in the metric. However, the two tensor degrees of freedom of the metric are the only physical ones (representing the two polarizations of the graviton) and they do not need to be generated by a matter source.^{||} Because the scalar fields do not generate the tensor perturbations, there is no difference between the treatment of these perturbations in multiple-field or in single-field inflation. Inflation enters only by way of the background quantities. Hence the results derived in this subsection are not new (see e.g. [33, 23, 34] and references therein), but we derive them here using the methods and definitions of [10].

The (ij) -component of the Einstein equation gives the well-known equation of motion for the tensor perturbation: $h''_{ij} + 2\mathcal{H}h'_{ij} - \Delta h_{ij} = 0$. Because h_{ij} is symmetric, transverse and traceless, it has only two independent components, which are represented by two constant polarization tensors e^A_{ij} , $A = 1, 2$, normalized as $e^A_{ij}e^{ijB} = \delta^{AB}$ and satisfying the same three properties as h_{ij} . The tensor perturbation is written as $h_{ij}(\eta, \mathbf{x}) = \sum_{A=1}^2 (2\kappa/a) \psi_A(\eta, \mathbf{x}) e^A_{ij}$, where the factor $2\kappa/a$ has been taken out to simplify the equation of motion and to obtain the correct normalization of the Lagrangean. After switching to Fourier modes we find the usual equation of motion for the mode functions $\psi_{A\mathbf{k}}(\eta)$:

$$\psi''_{A\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right) \psi_{A\mathbf{k}} = 0. \quad (18)$$

This equation is similar to equation (34) in [10] for the redefined gravitational potential $u_{\mathbf{k}}$, but without an inhomogeneous multiple-field term. An important difference between ψ_A and u is that the ψ_A represent two physical degrees of freedom and can be quantized directly, while u is not a physical degree of freedom, and had to be quantized indirectly by means of the scalar field degrees of freedom q . There is no coupling between the two different polarizations $A = 1$ and $A = 2$, as can be seen from (18). This means that it is not necessary to introduce a 2 by 2 matrix analogous to Q as we had to do for the quantization of q in (8) (as there is no coupling, this matrix would remain diagonal and can therefore be represented by a vector just as well). The Lagrangean of the system is

^{||} This is in contrast with vector perturbations, which cannot be generated by scalar-field perturbations either. One can easily derive from (2) that without a source term the $(0i)$ -component of the Einstein equation gives $\Delta S_i = 0$, so that the vector perturbations are zero. Moreover, even if there is a (non-inflation) vector source at some time, one finds from the (ij) -component of the Einstein equation that vector perturbations decay as a^{-2} during the expansion of the universe. Hence one only has to consider scalar and tensor perturbations in inflation theory, and not vector perturbations.

$L = \frac{1}{2}(\psi'_{A\mathbf{k}})^2 - \frac{1}{2}(k^2 - a''/a)(\psi_{A\mathbf{k}})^2$ and quantization is straightforward:

$$\hat{h}_{ij\mathbf{k}}(\eta) = \sum_{A=1}^2 \frac{2\kappa}{a} e_{ij}^A \left(\psi_{A\mathbf{k}}(\eta) \hat{a}_{A\mathbf{k}}^\dagger + \text{c.c.} \right), \quad (19)$$

with the creation and annihilation operators satisfying the usual commutation relations. Since the different Fourier modes, as well as the different polarizations, do not couple, we drop the subscripts A and \mathbf{k} for notational simplicity and consider one generic polarization and Fourier mode in the rest of this subsection.

In a way analogous to (but much simpler than) the treatment for the scalar perturbations in subsection 3.2 of [10] we can derive the initial conditions for ψ and the canonical momentum $\partial L/\partial\psi' = \psi'$ by using the canonical commutation relations between ψ and ψ' and the condition that the Hamiltonian does not contain any particle creation or annihilation terms initially, when k^2 is still much bigger than any other scale. This leads to the relations $\psi^*\psi' - \psi\psi'^* = i$ and $(\psi')^2 + k^2\psi^2 = 0$, with the solution $\psi_i = e^{i\alpha}/\sqrt{2k}$ and $\psi'_i = i\sqrt{k/2}e^{i\alpha}$. Here α is an arbitrary phase factor, which in a way completely analogous to the scalar case can be shown to be irrelevant to the physical correlator, just as the whole sub-horizon region, where ψ is simply oscillating, is irrelevant. Hence we take $\alpha = 0$ without loss of generality.

Realizing that $a''/a = \mathcal{H}^2(2 - \tilde{\epsilon})$ we see that the whole treatment in subsection 3.5 of [10] of the scalar case is easily applied to this case as well. In a sufficiently small interval around $\eta_{\mathcal{H}}$ (the time when $k = \mathcal{H}$) we find to first order in slow roll with $z = k\eta$:

$$\psi(z) = \sqrt{\frac{\pi}{4k}} \sqrt{z} H_{3/2+\tilde{\epsilon}_{\mathcal{H}}}^{(1)}(z) = -\frac{e^{i\pi\tilde{\epsilon}_{\mathcal{H}}}}{i\sqrt{2k}} [1 + \tilde{\epsilon}_{\mathcal{H}}(1 - \gamma - \ln 2)] \left(\frac{z}{z_{\mathcal{H}}} \right)^{-1-\tilde{\epsilon}_{\mathcal{H}}}, \quad (20)$$

where $H^{(1)}$ is a Hankel function and the expression after the last equals sign is only valid for $|z| \ll 1$.[¶] On the other hand, the solution in the super-horizon region where $k \ll \mathcal{H}$ is given by

$$\psi(z) = Ca + Da \int_{z_{\mathcal{H}}}^z \frac{dz'}{a^2}. \quad (21)$$

Integrating the expression $\mathcal{H}' = \mathcal{H}^2(1 - \tilde{\epsilon})$, with $\tilde{\epsilon} = \tilde{\epsilon}_{\mathcal{H}}$ taken constant (first-order slow-roll approximation), once with respect to conformal time we find $\mathcal{H}(\eta) = -[(1 - \tilde{\epsilon}_{\mathcal{H}})\eta]^{-1}$. Integrating once more, using $\mathcal{H} = (\ln a)'$, we find the following first-order approximation for $a(z)$ around $\eta = \eta_{\mathcal{H}}$: $a(z) = a_{\mathcal{H}}(z/z_{\mathcal{H}})^{-1-\tilde{\epsilon}_{\mathcal{H}}}$. Using the identification procedure described in [10] we see that again it is the leading-order term in the expansion in z of the Hankel function in the solution for ψ in the transition region that turns into the dominant solution for ψ in the super-horizon region, i.e. the C term in (21). From this we derive an expression for C : $C = (\sqrt{2k} a_{\mathcal{H}})^{-1} [1 + \tilde{\epsilon}_{\mathcal{H}}(1 - \gamma - \ln 2)]$. Here we omitted some unitary factors that are irrelevant to the calculation of the correlator of the tensor

[¶] Note that the change from the complex time dependence $e^{ik\eta}/\sqrt{2k}$ in the sub-horizon region, satisfying the quantum commutation relation $\psi^*\psi' - \psi\psi'^* = i$, to the real time dependence in (20) at the end of the transition region, satisfying the classical ($\hbar \rightarrow 0$) relation $\psi^*\psi' - \psi\psi'^* = 0$, is one way to see the quantum-to-classical transition taking place. For more information see [35, 36].

perturbations. As the D term in (21) rapidly decays compared to the C term, we do not have to determine D .

Using (19) we then obtain the final result for h_{ij} at later times (i.e. when we can neglect the D term) to first order in slow roll for super-horizon modes that crossed the horizon during slow-roll inflation:

$$\hat{h}_{ij\mathbf{k}} = \frac{\sqrt{2}\kappa}{k^{3/2}} H_{\mathcal{H}} [1 + \tilde{\epsilon}_{\mathcal{H}}(1 - \gamma - \ln 2)] \sum_{A=1}^2 e_{ij}^A \hat{a}_{A\mathbf{k}}^\dagger + \text{c.c.}, \quad (22)$$

where we used the identity $k = a_{\mathcal{H}} H_{\mathcal{H}}$. This result is also valid after inflation, as long as the mode \mathbf{k} remains super horizon. We see that this expression for h_{ij} is independent of time: the super-horizon h_{ij} is simply constant. Of course this could be seen directly from the equation of motion for h_{ij} for k^2 negligibly small (i.e. neglecting the Δh_{ij} term).

3. Inflation and the CMB

In this section (plus the appendix), expressions for the scalar (adiabatic, isocurvature and mixing) and tensor amplitudes and spectral indices and their derivatives are derived. The results are discussed, paying special attention to multiple-field effects. Moreover, a number of important relations between these quantities are derived, which can be used as observational tests to determine the presence of multiple fields during inflation.

3.1. Spectral quantities from inflation

Having determined the relevant scalar and tensor perturbation quantities at the time of recombination (see (9), (10) and (22)) we can now compute their correlators. We define the quantities $|\delta_{\mathbf{k}}^X|^2$ (X denoting adiabatic, isocurvature, mixing or tensor) as

$$|\delta_{\mathbf{k}}^{\text{ad}}|^2 \equiv \frac{2k^3}{9\pi^2} \langle \hat{\Phi}_{\mathbf{k}\text{ad}}^2 \rangle_{t_{\text{rec}}}, \quad |\delta_{\mathbf{k}}^{\text{iso}}|^2 \equiv \frac{2k^3}{9\pi^2} \langle \hat{\Phi}_{\mathbf{k}\text{iso}}^2 \rangle_{t_{\text{rec}}}, \quad |\delta_{\mathbf{k}}^{\text{tens}}|^2 \equiv \frac{2k^3}{9\pi^2} \langle \hat{h}_{ij\mathbf{k}} \hat{h}_{\mathbf{k}}^{ij} \rangle_{t_{\text{rec}}},$$

$$|\delta_{\mathbf{k}}^{\text{mix}}|^2 \equiv \frac{2k^3}{9\pi^2} \left(\langle \hat{\Phi}_{\mathbf{k}\text{iso}} \hat{\Phi}_{\mathbf{k}\text{ad}} \rangle_{t_{\text{rec}}} + \langle \hat{\Phi}_{\mathbf{k}\text{ad}} \hat{\Phi}_{\mathbf{k}\text{iso}} \rangle_{t_{\text{rec}}} \right). \quad (23)$$

Unfortunately there are different conventions in the literature regarding the normalization factor. The normalization used here corresponds with [37, 38], in which papers the relation between this amplitude and the observations is explained extensively. The normalization factor in, for example, [34], which was used in the table with numerical values in our paper [10] as well, is $9/4$ times larger.

If $|\delta_{\mathbf{k}}^X|^2$ depends only weakly on k , the following approximation can be made:

$$|\delta_{\mathbf{k}}^X|^2 = |\delta_{\mathbf{k}_0}^X|^2 \left(\frac{k}{k_0} \right)^{\tilde{n}_X}, \quad (24)$$

with k_0 a certain reference scale. $|\delta_{\mathbf{k}_0}^X|^2$ and \tilde{n}_X are two constants, called the amplitude of the CMB power spectrum and the spectral index, respectively. This approximation holds good over a wide range of k if \tilde{n}_X is close to zero, which is the case for slow-roll inflation (see later). Note that originally the spectral index was defined as $n_X = \tilde{n}_X + 1$,

except for the tensor perturbations where the above definition was indeed used. As this was a rather unfortunate source of confusion, newer papers usually adopt the definition given here, but to avoid confusion we have added the tilde.

With these definitions we find for the amplitudes the following expressions, valid up to and including first order in slow roll:

$$|\delta_{\mathbf{k}}^{\text{ad}}|^2 = \frac{\kappa^2}{50\pi^2} \frac{H_{\mathcal{H}}^2}{\tilde{\epsilon}_{\mathcal{H}}} \left[(1 - 2\tilde{\epsilon}_{\mathcal{H}})(1 + U_{Pe}^T U_{Pe}) + 2B \left(2\tilde{\epsilon}_{\mathcal{H}} + \tilde{\eta}_{\mathcal{H}}^{\parallel} + 2\tilde{\eta}_{\mathcal{H}}^{\perp} e_2^T U_{Pe} + U_{Pe}^T \delta_{\mathcal{H}} U_{Pe} \right) \right], \quad (25)$$

$$|\delta_{\mathbf{k}}^{\text{iso}}|^2 = \frac{1}{36} \frac{\kappa^2}{50\pi^2} \frac{H_{\mathcal{H}}^2}{\tilde{\epsilon}_{\mathcal{H}}} \left[(1 - 2\tilde{\epsilon}_{\mathcal{H}}) V_e^T V_e + 2B V_e^T \delta_{\mathcal{H}} V_e \right], \quad (26)$$

$$|\delta_{\mathbf{k}}^{\text{mix}}|^2 = \frac{1}{6} \frac{\kappa^2}{50\pi^2} \frac{H_{\mathcal{H}}^2}{\tilde{\epsilon}_{\mathcal{H}}} \left[(1 - 2\tilde{\epsilon}_{\mathcal{H}}) U_{Pe}^T V_e + 2B \left(\tilde{\eta}_{\mathcal{H}}^{\perp} e_2^T V_e + U_{Pe}^T \delta_{\mathcal{H}} V_e \right) \right], \quad (27)$$

$$|\delta_{\mathbf{k}}^{\text{tens}}|^2 = \frac{400}{9} \frac{\kappa^2}{50\pi^2} H_{\mathcal{H}}^2 \left[1 + 2(B - 1)\tilde{\epsilon}_{\mathcal{H}} \right]. \quad (28)$$

Here U_{Pe} , V_e and δ are defined in (12) and (11), and $B \equiv 2 - \gamma - \ln 2 \approx 0.7296$. Note that $U_{Pe}^T U_{Pe}$ and $V_e^T V_e$ can be of order unity and hence the leading-order expressions include the $(1 + U_{Pe}^T U_{Pe})$, the $V_e^T V_e$, and the $U_{Pe}^T V_e$ factors, respectively. The reader interested in more detail about the derivation is referred to [10, 22].

The spectral indices \tilde{n}_X can be calculated from the expressions for $|\delta_{\mathbf{k}}^X|^2$ above:

$$\tilde{n}_X \equiv \left. \frac{d \ln |\delta_{\mathbf{k}}^X|^2}{d \ln k} \right|_{\mathbf{k}=\mathbf{k}_0} = \frac{d \ln |\delta_{\mathbf{k}}^X|^2}{dt_{\mathcal{H}}} \frac{dt_{\mathcal{H}}}{d \ln k} = \frac{d \ln |\delta_{\mathbf{k}}^X|^2}{dt_{\mathcal{H}}} \frac{1}{H_{\mathcal{H}}(1 - \tilde{\epsilon}_{\mathcal{H}})}. \quad (29)$$

Here we omitted the explicit $\mathbf{k} = \mathbf{k}_0$ from the last two steps, but of course it should be applied there as well. In the last step we used $dt_{\mathcal{H}}/d \ln k = (d \ln k/dt_{\mathcal{H}})^{-1}$ and $\mathcal{H}_{\mathcal{H}} = k$. To work out this expression we need the derivatives of U_{Pe} and V_e with respect to $t_{\mathcal{H}}$:

$$\begin{aligned} \frac{dU_{Pe}^T}{dt_{\mathcal{H}}} &= \frac{1}{2} \frac{\dot{\tilde{\epsilon}}_{\mathcal{H}}}{\tilde{\epsilon}_{\mathcal{H}}} U_{Pe}^T - 2H_{\mathcal{H}} \tilde{\eta}_{\mathcal{H}}^{\perp} e_2^T + \frac{\dot{a}_{\mathcal{H}}}{a_{\mathcal{H}}} U_{Pe}^T + U_{Pe}^T Q_{\mathcal{H}} (Q_{\mathcal{H}}^{-1})^{\bullet} \\ &= H_{\mathcal{H}} U_{Pe}^T \left(2\tilde{\epsilon}_{\mathcal{H}} + \tilde{\eta}_{\mathcal{H}}^{\parallel} - \delta_{\mathcal{H}} \right) - 2H_{\mathcal{H}} \tilde{\eta}_{\mathcal{H}}^{\perp} e_2^T, \end{aligned} \quad (30)$$

$$\frac{dV_e^T}{dt_{\mathcal{H}}} = H_{\mathcal{H}} V_e^T \left(2\tilde{\epsilon}_{\mathcal{H}} + \tilde{\eta}_{\mathcal{H}}^{\parallel} - \delta_{\mathcal{H}} \right). \quad (31)$$

Here we used $(Q_{\mathcal{H}}^{-1})^{\bullet} = -Q_{\mathcal{H}}^{-1} \dot{Q}_{\mathcal{H}} Q_{\mathcal{H}}^{-1}$ and $\dot{Q}_{\mathcal{H}} = H_{\mathcal{H}}(1 - \tilde{\epsilon}_{\mathcal{H}} + \delta_{\mathcal{H}})Q_{\mathcal{H}}$. The latter result follows from the fact that $Q \propto \eta^{-1-\delta_{\mathcal{H}}}$ near $\eta_{\mathcal{H}}$ (see e.g. equation (63) in [10]) and using the expression for $\mathcal{H}(\eta)$ in the text below (21). The final results to leading (first) order in slow roll are

$$[\tilde{n}_{\text{ad}}]_{\text{l.o.}} = -2 \frac{2\tilde{\epsilon}_{\mathcal{H}} + \tilde{\eta}_{\mathcal{H}}^{\parallel} + 2\tilde{\eta}_{\mathcal{H}}^{\perp} e_2^T U_{Pe} + U_{Pe}^T \delta_{\mathcal{H}} U_{Pe}}{1 + U_{Pe}^T U_{Pe}}, \quad (32)$$

$$[\tilde{n}_{\text{iso}}]_{\text{l.o.}} = -2 \frac{V_e^T \delta_{\mathcal{H}} V_e}{V_e^T V_e}, \quad (33)$$

$$[\tilde{n}_{\text{mix}}]_{\text{l.o.}} = -2 \frac{\tilde{\eta}_{\mathcal{H}}^{\perp} e_2^T V_e + U_{Pe}^T \delta_{\mathcal{H}} V_e}{U_{Pe}^T V_e}, \quad (34)$$

$$[\tilde{n}_{\text{tens}}]_{\text{l.o.}} = -2\tilde{\epsilon}_{\mathcal{H}}, \quad (35)$$

where l.o. means leading order. The result for the adiabatic spectral index was previously derived in our paper [39]. When comparing (25)–(28) and (32)–(35), we see that we can rewrite the spectral amplitudes to first order in slow roll in terms of the leading-order expressions:

$$|\delta_{\mathbf{k}}^X|^2 = |\delta_{\mathbf{k}}^X|_{\text{l.o.}}^2 (1 + [\tilde{n}_{\text{tens}}]_{\text{l.o.}} - B[\tilde{n}_X]_{\text{l.o.}}). \quad (36)$$

This means that using (29) we can immediately extend our results for the spectral indices to include the next-to-leading-order (i.e. second-order) terms:

$$\tilde{n}_X = [\tilde{n}_X]_{\text{l.o.}} (1 + \tilde{\epsilon}_{\mathcal{H}}) + \left[\frac{d\tilde{n}_{\text{tens}}}{d \ln k} \right]_{\text{l.o.}} - B \left[\frac{d\tilde{n}_X}{d \ln k} \right]_{\text{l.o.}}. \quad (37)$$

The $d\tilde{n}_X/d \ln k$ are given in the appendix.

3.2. Discussion

We continue with a discussion of the results derived in the previous subsection, leading to several important conclusions. Some of these were already given in [10], but are repeated here for completeness' sake.

We see that the \tilde{n}_X contain only slow-roll terms, and thus the $|\delta_{\mathbf{k}}^X|^2$ do indeed depend on k very weakly. In other words, the spectrum predicted by slow-roll inflation is nearly scale-invariant. Of course if $V_e = 0$, there are no isocurvature perturbations and \tilde{n}_{iso} and \tilde{n}_{mix} become meaningless. The derivatives of the spectral indices $d\tilde{n}_X/d \ln k$ are second order in slow roll. This might be too small according to recent WMAP data [7, 9], where a $d\tilde{n}_X/d \ln k$ of the same order of magnitude as \tilde{n}_X is claimed, leading to a problem for slow-roll models of inflation in general. However, as explained in [8], this result is only obtained when combining WMAP with other data sets, especially Lyman- α data. Hence this result is as yet controversial, with several authors [40, 41, 42, 43] claiming that a constant \tilde{n}_X is perfectly consistent with the data.

The explicit multiple-field terms in the amplitudes and the spectral indices are the contributions of the terms U_{Pe} and V_e , which are absent in the single-field case (setting them equal to zero we obtain the well-known single-field results, see e.g. [34]). There are no isocurvature and mixing contributions in the single-field case. Since both U_{Pe} and V_e are to a large extent determined by $\tilde{\eta}^\perp$ (see their definitions in (12)), we can draw the important conclusion that the behaviour of $\tilde{\eta}^\perp$ during the last 60 e-folds of inflation is crucial in order to determine whether multiple-field effects are important. On the other hand, the fact that U_{Pe} depends on $\tilde{\eta}^\perp$ should not be taken as an indication that it is a first-order quantity: as was shown in the example in [10] U_{Pe} can be of zeroth-order importance (because of the integration interval).

The fact that the entropy perturbations act as sources for the adiabatic perturbation (the U_{Pe} terms) naturally leads to correlations between adiabatic and isocurvature perturbations (described by the mixing amplitude), as had been realized before (see e.g. [44, 45, 46]). Note that even if $U_{Pe} = 0$ there is still one other term in the mixing amplitude, although merely of first order in slow roll. Only if $\tilde{\eta}_{\mathcal{H}}^\perp$ vanishes as well, the

correlations are completely absent. However, at least in the context of slow roll the situation where $U_{Pe} = 0$ while $\tilde{\eta}_{\mathcal{H}}^\perp, V_e \neq 0$ is not possible, because the $e_2^T Q/a$ under the integral in the definition of U_{Pe} cannot change sign. Anyhow, if $\tilde{\eta}^\perp = 0$ everywhere during the last 60 e-folds, there are certainly no correlations. (The authors of [45] studied the two-field case and found the derivative of the angle that parametrizes the influence of the second field on the background trajectory to be the relevant parameter. In the two-field limit this parameter corresponds with $\tilde{\eta}^\perp$, but our result is valid for an arbitrary number of fields. A general discussion of two-field models has recently been given in [17].)

The gravitational potential only depends on the *total* entropy perturbation \tilde{S} , independently of the total number of fields and the actual number of independent entropy perturbations. In our basis \tilde{S} depends directly on the e_2 component of q only (see the expression for V_e in (12) or (17)). (Of course the other components of q influence the equation of motion for q_2 and cannot be neglected in general.) This is the main reason why we only consider this total entropy perturbation and a total isocurvature amplitude in this paper. However, for a complete understanding of the CMB power spectrum it will probably be necessary to investigate the individual entropy perturbations as well.⁺

Using the concept of slow roll for the perturbations the quantity U_{Pe} can be rewritten in terms of background quantities only, as was discussed in [10] (see equation (69) in that paper). Because slow roll was used in the derivation, that expression is in principle not valid at the very end of inflation. (Note that until this point slow roll was only used during a small transition region around the time of horizon crossing deep within the inflationary era.) If it does indeed give a bad approximation for U_{Pe} , for example if $\tilde{\eta}^\perp$ grows very large, a more careful treatment of the transition at the end of inflation is necessary. However, in other cases the contribution to the integral near the end of inflation can be negligible, for example if $\tilde{\eta}^\perp$ goes sufficiently rapidly to zero. In those cases the details at the end of inflation are unimportant for the adiabatic amplitude (25). An example of this latter case was discussed in [10]. Unfortunately V_e depends very much on the details of the transition at the end of inflation, so that for an accurate calculation a model of this transition has to be assumed. It also depends on the behaviour of \tilde{S} during radiation and matter domination, as shown in the derivation of (17), but this aspect is investigated in section 4. However, one can still draw the conclusion that, if $\tilde{\eta}^\perp$ goes to zero at the end of inflation, the isocurvature perturbations are expected to be negligible compared with the adiabatic one (neglecting possible amplifying mechanisms during the transition and preheating).

Compared with the scalar amplitudes, an overall factor of $1/\tilde{\epsilon}_{\mathcal{H}}$ is missing in the

⁺ For example, one can prove that if there is not only baryonic matter, radiation and cold dark matter, but hot dark matter and/or quintessence as well, their individual entropy perturbations enter the expression for the Sachs-Wolfe effect (see [22]). An investigation of the individual entropy perturbations in a model with photons, baryons, neutrinos, and cold dark matter was given in [49]. The same authors investigated the detectability of these entropy perturbations in upcoming experiments in [50] and concluded that we will have to wait for the Planck satellite [51] for sufficient discriminating power.

expression for the tensor amplitude, showing that the tensor contribution to the CMB from slow-roll inflation will generically be smaller than the scalar one. Moreover, this means that the tensor amplitude depends only on $H_{\mathcal{H}}^2$ (to leading order), thus allowing a direct determination of the inflationary energy scale. Another important difference between the tensor and scalar quantities is that the tensor ones do not depend on multiple-field terms. As explained in subsection 2.3 this is because the tensor perturbations are not generated by the scalar-field perturbations and hence do not see the difference between multiple- and single-field inflation. It is this fact that allows us to derive, in the following subsection, an important relation that can be used as an observational test for the importance of multiple-field effects.

3.3. Observational tests and consistency relations

From the expressions for the spectral amplitudes and indices ((25)–(28) and (32)–(35)) we can derive some important relations between these quantities. Note that we only use the general expressions for these amplitudes and indices in the derivations, not the specific form of, for example, V_e . Therefore the results are valid very generally, and do not depend on the details of the evolution of the entropy perturbations after inflation (as was explained in the derivation of (17), these only change the definition of V_e).

We define the tensor to scalar ratio r as $r \equiv |\delta_{\mathbf{k}}^{\text{tens}}|^2/|\delta_{\mathbf{k}}^{\text{ad}}|^2$. Then we find the following relation to leading order:

$$r = \frac{400}{9} \frac{\tilde{\epsilon}_{\mathcal{H}}}{1 + U_{Pe}^T U_{Pe}} = -\frac{200}{9} \frac{\tilde{n}_{\text{tens}}}{1 + U_{Pe}^T U_{Pe}} \Rightarrow U_{Pe}^T U_{Pe} = -1 - \frac{200}{9} \frac{\tilde{n}_{\text{tens}}}{r}. \quad (38)$$

This expression is a very important result: it gives a relation between observable quantities and the length of the vector U_{Pe} that encodes the effects of multiple fields. Once observations have become good enough to determine the tensor to scalar ratio r and the tensor spectral index \tilde{n}_{tens} independently, this relation offers an observational test to check whether multiple-field effects are important or not.

Using (36) and (37) we can extend this result to next-to-leading order:

$$U_{Pe}^T U_{Pe} = -1 - \frac{200}{9} \frac{\tilde{n}_{\text{tens}}}{r} \left[1 - (B - \tfrac{1}{2})\tilde{n}_{\text{tens}} + B\tilde{n}_{\text{ad}} + (B - 1) \frac{1}{\tilde{n}_{\text{tens}}} \frac{d\tilde{n}_{\text{tens}}}{d \ln k} \right]. \quad (39)$$

This does not change the above conclusion: it is still a relation that allows the multiple-field quantity $U_{Pe}^T U_{Pe}$ to be determined from observations.* We can also (in principle) determine the other multiple-field quantities $V_e^T V_e$ and $U_{Pe}^T V_e$ from the observations:

$$V_e^T V_e = -800 \frac{\tilde{n}_{\text{tens}}}{|\delta_{\mathbf{k}}^{\text{tens}}|^2/|\delta_{\mathbf{k}}^{\text{iso}}|^2}, \quad U_{Pe}^T V_e = -\frac{400}{3} \frac{\tilde{n}_{\text{tens}}}{|\delta_{\mathbf{k}}^{\text{tens}}|^2/|\delta_{\mathbf{k}}^{\text{mix}}|^2}, \quad (40)$$

with similar extensions to the next order as in (39).

* There is one other effect that might lead to corrections to this formula: non-vacuum initial states caused by trans-Planckian physics [52]. However, this effect is expected to be small: deviations from the vacuum initial state should not be too large, otherwise the particle background dominates over the potential energy of the inflaton and there is no (standard) inflation [53]. For a recent review of trans-Planckian effects see [54].

Next to the above results valid for an arbitrary number of fields, we can find some further results in the case of one or two fields only. These are the so-called consistency relations, which are based on the fact that in those two cases we have more observational quantities than inflationary parameters. In the single-field case the unknowns are $H_{\mathcal{H}}$, $\tilde{\epsilon}_{\mathcal{H}}$ and $\tilde{\eta}_{\mathcal{H}}^{\parallel}$, while the scalar and tensor amplitudes and indices give four observational quantities. Hence we can express one of these in terms of the others: a consistency relation. Conventionally this is written as follows (valid to leading order) [55, 37, 56]:

$$r = -\frac{200}{9} \tilde{n}_{\text{tens}} \quad (\text{one field}), \quad (41)$$

a relation that follows immediately from the single-field limit of (38).

In the two-field case, there are four more inflationary parameters: $\tilde{\eta}_{\mathcal{H}}^{\perp}$, U_{Pe} and V_e (both are vectors, but in the two-field case there is only one non-zero component), and $\delta_{\mathcal{H}}$ (this is a matrix, but in the two-field case there is only one unknown component). As there are also four more observational (in principle at least) quantities, there is still one consistency relation, which can be written as (again only valid to leading order):

$$r = -\frac{200}{9} \tilde{n}_{\text{tens}} (1 - r_{\text{mix}}^2) \quad (\text{two fields}). \quad (42)$$

Here r_{mix}^2 is defined as

$$r_{\text{mix}}^2 \equiv \frac{|\delta_{\mathbf{k}}^{\text{mix}}|^4}{|\delta_{\mathbf{k}}^{\text{ad}}|^2 |\delta_{\mathbf{k}}^{\text{iso}}|^2} = \frac{(U_{Pe}^T V_e)^2}{(1 + U_{Pe}^T U_{Pe}) V_e^T V_e}, \quad (43)$$

where the second expression is valid to leading order. In the two-field case where the vectors U_{Pe} and V_e have only one non-zero component (denoted by U and V , respectively), the V drops out and one has $r_{\text{mix}}^2 = U^2/(1 + U^2)$, which leads to the expression in (42). This two-field consistency relation was also derived in a different way in [15, 16].[‡] Of course the two-field consistency relation reduces to the single-field one in the appropriate limit because r_{mix} is zero then. In the case of three or more fields the number of inflationary parameters increases (with four in the case of three fields: one additional component in U_{Pe} and V_e and two in $\delta_{\mathcal{H}}$) without an increase in the number of observational quantities (at least when considering only the total isocurvature perturbation). Hence there are no consistency relations in those cases. For some consistency relations concerning the $d\tilde{n}_X/d \ln k$ see the appendix.

To conclude this section: assuming that we will be able to measure the scalar and tensor amplitudes and indices with sufficient accuracy in the near future (which of course implicitly assumes that they are large enough to be measured), we can then first use equation (38) (or (39)) to check if multiple-field effects are significant at all. Next we can use the consistency relation (42) to distinguish between the cases of two or more fields. Note that an accurate measurement of the isocurvature and mixing quantities is only necessary for the second step, not for the first. An accurate measurement of the

[‡] In [15] a second consistency relation for the two-field case was derived as well. However, this relation is not valid in general, as was also pointed out in [16], but only if the inflation model satisfies certain specific additional conditions (namely that the (22)-component of the matrix $\delta_{\mathcal{H}}$ is equal to $2\tilde{\epsilon}_{\mathcal{H}} + \tilde{\eta}_{\mathcal{H}}^{\parallel}$).

tensor amplitude and spectral index is, however, essential for both steps (for a recent study on detectability issues see [57]).

4. Entropy perturbations

In the derivation of the isocurvature perturbation (10), it was first assumed that the total entropy perturbation \tilde{S} is constant during radiation and matter domination on super-horizon scales, leading to equation (12) for V_e . We then considered the possibility that \tilde{S} is not constant, finding that (10) is still correct, but now with a more general expression (17) for V_e . In this section, we investigate the total entropy perturbation during radiation and matter domination to determine its evolution with time.

4.1. Total entropy perturbation

We consider a universe filled with an arbitrary number N of energy components, labelled by the subscript i . The different components each have a pressure p_i and an energy density ρ_i , as well as pressure and density perturbations. For the total pressure $p = \sum_i p_i$ and the total energy density $\rho = \sum_i \rho_i$ we have

$$w \equiv \frac{p}{\rho}, \quad c_s^2 \equiv \frac{\dot{p}}{\dot{\rho}}, \quad \dot{\rho} + 3H\rho(1+w) = 0, \quad \dot{w} = -3H(c_s^2 - w)(1+w). \quad (44)$$

The first two expressions are definitions (w is called the equation of state parameter and c_s^2 the sound velocity), the third one is the energy-momentum conservation condition $D_\mu T^\mu_0 = 0$, and the last equation follows by writing $\dot{w} = (\dot{p}/\dot{\rho} - p/\rho)\dot{\rho}/\rho$. For the individual components we define analogous quantities: $w_i \equiv p_i/\rho_i$ and $c_i^2 \equiv \dot{p}_i/\dot{\rho}_i$ (note that in general $w \neq \sum_i w_i$ and $c_s^2 \neq \sum_i c_i^2$) and find

$$\dot{\rho}_i + 3H\rho_i(1+w_i) = 3HC_i, \quad \dot{w}_i = -3H(c_i^2 - w_i) \left[(1+w_i) - \frac{C_i}{\rho_i} \right], \quad (45)$$

with C_i a measure of the interactions between the different components satisfying $\sum_i C_i = 0$. In the following we make two assumptions regarding the separate components, which for the rest are completely arbitrary:

- (i) All components behave as ideal fluids with a constant w_i .
- (ii) There are no interactions: $C_i = 0$ for all i .

From equation (45) we see that this automatically means that $c_i^2 = w_i$ (the square in c_i^2 is just convention; c_i^2 can be negative). Moreover, a constant w_i also means that $\delta p_i/\delta \rho_i = w_i$.

Let us remark briefly on the assumption of no interactions with regard to a real model. One can think of the following situation. Of the multiple scalar fields during inflation, one has decayed to all the Standard Model particles, while the other fields have decayed to various kinds of dark matter. Then there are by construction no entropy perturbations between the Standard Model components on super-horizon scales and the absence or presence of interactions here is irrelevant. There are only entropy

perturbations between the dark matter and Standard Model components and between the dark matter components themselves, where the assumption of no interactions seems quite plausible. It is usually assumed (see e.g. [58, 44]) that including interactions will have the effect of wiping out the isocurvature perturbations. However, it will be interesting to check this more carefully in the near future.

To rewrite \tilde{S} (defined as $\tilde{S} \equiv \frac{1}{4}(\delta p - c_s^2 \delta \rho)/(p - c_s^2 \rho)$) we need some auxiliary results. In the first place we have that $p = \sum_i w_i \rho_i$, so that

$$c_s^2 = \frac{\dot{p}}{\dot{\rho}} = \sum_i \frac{\rho_i(1 + w_i)}{\rho(1 + w)} w_i, \quad (46)$$

where we used (45) and (44) for $\dot{\rho}_i$ and $\dot{\rho}$. Using this result the numerator of \tilde{S} is rewritten as follows:

$$\begin{aligned} \delta p - c_s^2 \delta \rho &= \sum_k (w_k - c_s^2) \delta \rho_k \\ &= \frac{1}{\rho(1 + w)} \sum_{k,l} \rho_l (1 + w_l) (w_k - w_l) \delta \rho_k \\ &= \frac{1}{\rho(1 + w)} \frac{1}{2} \sum_{k,l} \rho_k \rho_l (1 + w_k) (1 + w_l) (w_k - w_l) S_{kl} \end{aligned} \quad (47)$$

with

$$S_{kl} \equiv \frac{\delta \rho_k}{\rho_k(1 + w_k)} - \frac{\delta \rho_l}{\rho_l(1 + w_l)}. \quad (48)$$

In the last step of (47) we symmetrized the expression in k and l . Completely analogously we find for the denominator

$$p - c_s^2 \rho = \sum_k (w_k - c_s^2) \rho_k = -\frac{1}{\rho(1 + w)} \frac{1}{2} \sum_{k,l} \rho_k \rho_l (w_k - w_l)^2. \quad (49)$$

Our final result for the total entropy perturbation is then

$$\tilde{S} = -\frac{1}{4} \frac{\sum_{k,l} \rho_k \rho_l (1 + w_k) (1 + w_l) (w_k - w_l) S_{kl}}{\sum_{k,l} \rho_k \rho_l (w_k - w_l)^2}. \quad (50)$$

The S_{kl} are the individual entropy (or isocurvature) perturbations [59, 60, 45] (see also [61]). They are antisymmetric in k and l , and in addition one has $S_{kl} = S_{km} - S_{lm}$. This means that the matrix S_{kl} contains $(N - 1)$ independent elements. One can take a single reference component 0 and define $S_k \equiv S_{k0}$, so that $S_{kl} = S_k - S_l$, with of course $S_0 = 0$. Hence if we have a system of N components, there are in general 1 adiabatic and $(N - 1)$ entropy perturbations. The combination \tilde{S} of these $(N - 1)$ entropy perturbations that enters as the source term into the equation for Φ is what we call the total entropy perturbation. In the case of inflation with N fields, the adiabatic perturbation corresponds with the e_1 direction in our basis, while the perturbations in the $(N - 1)$ other directions are isocurvature perturbations. The total entropy perturbation \tilde{S} then corresponds exactly with the e_2 direction in our basis, see (15). The entropy perturbations S_{kl} are gauge-invariant by definition in the absence of interactions,

see e.g. [60], and can even be defined in such a way that they are gauge-invariant when interactions are included, see [61]. The total entropy perturbation \tilde{S} defined in (14) is always gauge-invariant.

Working out (50) in the case of an arbitrary number of matter components labelled by m (e.g. baryons or cold dark matter with $w_m = 0$) and an arbitrary number of radiation components labelled by r (e.g. photons or hot dark matter with $w_r = 1/3$), we obtain

$$\tilde{S} = \frac{\sum_{m,r} \rho_m \rho_r S_{mr}}{\sum_{m,r} \rho_m \rho_r} = \frac{\sum_m \rho_m S_m}{\sum_m \rho_m} - \frac{\sum_r \rho_r S_r}{\sum_r \rho_r}. \quad (51)$$

In the last step we have singled out one of the radiation components, for example the photons γ , as the reference component, so that $S_m = S_{m\gamma}$ and $S_r = S_{r\gamma}$. (One could just as well choose one of the matter components as reference. The only difference with (51) is then an overall minus sign.) In the case of a simple two-component system consisting of photons γ and one cold dark matter component C , which is the case usually considered in inflationary literature (see e.g. [58, 44, 45, 15]), this result simplifies to $\tilde{S} = S_C = (\delta\rho_C/\rho_C) - \frac{3}{4}(\delta\rho_\gamma/\rho_\gamma)$.

4.2. Time dependence of \tilde{S}

Next we derive an expression for the time derivative of \tilde{S} . First we need an equation of motion for $\delta\rho_i$. This equation can be derived by working out the condition $D_\mu T^\mu_0 = 0$ to first order in the perturbations. However, as we are only interested in the super-horizon modes, it is simpler to use the method of varying the background equation. (A description of this method can be found in e.g. subsection 3.4 of [10] and references therein.) From (45), with the two assumptions of ideal fluids without interactions, we then find

$$\delta\dot{\rho}_i + 3H\delta\rho_i(1 + w_i) - 3\dot{\Phi}\rho_i(1 + w_i) = 0, \quad (52)$$

using $\delta H = (\delta \ln a)^\cdot = -\dot{\Phi}$. Equation (52) can also be found in [60]. From this result together with (45) we easily derive that

$$\dot{S}_{kl} = \frac{\delta\rho_k}{\rho_k(1 + w_k)} \left(\frac{\delta\dot{\rho}_k}{\delta\rho_k} - \frac{\dot{\rho}_k}{\rho_k} \right) - (k \leftrightarrow l) = 0. \quad (53)$$

When differentiating \tilde{S} , given in (50), this means that the time dependence is completely determined by the background quantities. We find after a long calculation

$$\begin{aligned} \dot{\tilde{S}} &= \frac{3}{4}H \frac{\sum_{i,j,k,l} [(w_k + w_l) - (w_i + w_j)](w_k - w_l)(w_i - w_j)^2(1 + w_k)(1 + w_l)\rho_i\rho_j\rho_k\rho_l S_{kl}}{\left(\sum_{i,j} (w_i - w_j)^2 \rho_i \rho_j \right)^2} \\ &= \frac{3}{4}H \frac{\sum_{i,j,k} w_i(w_i - w_j)[w_k^2 - w_i w_k + w_i w_j]\rho_i \rho_j \delta\rho_k}{(1 + w)(w - c_s^2)^2 \rho^3}. \end{aligned} \quad (54)$$

The second form of the result comes about after inserting the definition of S_{kl} (48), substantial index manipulation in the numerator and using (49) in the denominator. It is more compact, but in some ways the first expression is more useful. From the

first expression we can see immediately that the time derivative of \tilde{S} will be zero in the case of components with only two different values of w_i (one cannot make all three of $[(w_k + w_l) - (w_i + w_j)]$, $(w_i - w_j)$ and $(w_k - w_l)$ unequal to zero in that case). Hence we can draw the important conclusion that in a universe consisting only of matter and radiation components, \tilde{S} remains constant on super-horizon scales, irrespective of how many kinds of matter and radiation there are (provided that the two assumptions of constant w_i and no interactions are valid). This includes a universe with an arbitrary number of hot and cold dark matter components.

Let us also give the result for $\dot{\tilde{S}}$ in the case that we relax the second assumption, that is, if we include interactions. The sound velocity and total entropy perturbation are then given by

$$c_s^2 = \sum_i \frac{\rho_i(1 + w_i) - C_i}{\rho(1 + w)} w_i, \quad \tilde{S} = \frac{1}{4} \frac{\sum_{k,l} (1 + w_l)(w_k - w_l) \rho_l \delta \rho_k + w_l C_l \delta \rho_k}{\sum_{k,l} (1 + w_l)(w_k - w_l) \rho_k \rho_l + w_l C_l \rho_k}. \quad (55)$$

(Since \dot{S}_{kl} is no longer zero, it is not useful to rewrite \tilde{S} in terms of the individual entropy perturbations in this case.) After a long calculation with substantial index manipulation, during which one should keep in mind that $\sum_i C_i = 0$, we finally obtain:

$$\begin{aligned} \dot{\tilde{S}} = \frac{3}{4} H [(1 + w)(w - c_s^2)^2 \rho^3]^{-1} \sum_{i,j,k} \Big\{ & w_i(w_i - w_j)(w_k^2 - w_i w_k + w_i w_j) \rho_i \rho_j \delta \rho_k \\ & + w_j(w_i - w_k)(1 + 2w_i - w_j + w_k) \rho_i C_j \delta \rho_k - w_i w_j C_i C_j \delta \rho_k \\ & + w_j(w_i - w_k) \rho_i \frac{\dot{C}_j}{3H} \delta \rho_k - w_i w_k (w_i - w_j) \rho_i \rho_j \delta C_k + w_j w_k \rho_i C_j \delta C_k \Big\}. \end{aligned} \quad (56)$$

A further treatment of the interacting case is postponed to a future publication.

Next we consider what happens in the case that there are components with three different values of w_i , for example matter, radiation and a cosmological constant, again neglecting interactions. With more than two different components \tilde{S} is in general not constant. Even though (54) gives an expression for its time derivative, it is in general difficult to solve explicitly because the time dependence of the perturbations $\delta \rho_k$ is not trivial to determine: from (52) we see that it depends on $\dot{\Phi}$ so that the equations for \tilde{S} and Φ have to be solved together. However, it turns out that in the special case of components with three different values of w_i , the time dependence of \tilde{S} is determined by the background energy densities only and does not depend on Φ .

The main idea is that we take another time derivative and calculate $\ddot{\tilde{S}}$. For three components we find that the numerator of \tilde{S} in (54) can be written as

$$g(w_1, w_2, w_3) \rho_1 \rho_2 \delta \rho_3 + g(w_3, w_1, w_2) \rho_1 \rho_3 \delta \rho_2 + g(w_2, w_3, w_1) \rho_2 \rho_3 \delta \rho_1, \quad (57)$$

with $g(a, b, c) = (a - b)^2(c^2 + ab - c(a + b))$. Taking the time derivative of this expression using (45) and (52) we find simply $-3H(3 + w_1 + w_2 + w_3)$ times the same expression (57) (the $\dot{\Phi}$ terms from (52) exactly cancel, which can even be proved to be true for an arbitrary number of components). Hence we conclude that in the three-component case $\dot{\tilde{S}} \propto \ddot{\tilde{S}}$, where the proportionality factor only depends on the background energy densities, not on the energy density perturbations. This means that, given initial

conditions for \tilde{S} and $\dot{\tilde{S}}$, one can explicitly solve for $\tilde{S}(t)$, without needing solutions for $\delta\rho_i(t)$ and $\Phi(t)$. For four or more components this is no longer true.

As an explicit example we consider a system with matter m , radiation r and a cosmological constant Λ , where $w_m = 0$, $w_r = 1/3$, and $w_\Lambda = -1$. Then we have

$$\tilde{S}_{,NN} = 4\tilde{S}_{,N} \frac{\rho_m^2 \rho_r + \rho_m \rho_r^2 - 9\rho_m^2 \rho_\Lambda - 16\rho_r^2 \rho_\Lambda - 23\rho_m \rho_r \rho_\Lambda}{(\rho_m + \frac{4}{3}\rho_r)(\rho_m \rho_r + 9\rho_m \rho_\Lambda + 16\rho_r \rho_\Lambda)}. \quad (58)$$

Even though this is not during inflation, it turns out to be convenient to use the number of e-folds N as time variable to remove the H and its derivative. We choose N to be zero at the present time and negative before that. The functions $\rho_i(N)$ can be determined from (45): $\rho_i(N) = \Omega_i \rho_c \exp(-3(1+w_i)N)$, where $\Omega_i \equiv \rho_i(0)/\rho_c$ is the present density parameter of component i and $\rho_c \equiv 3H_0^2/\kappa^2$ is the present critical density, which drops out of the equations, however. Using data from WMAP [7] we have

$$\Omega_m = 0.3, \quad \Omega_r = 5 \cdot 10^{-5}, \quad \Omega_\Lambda = 0.7, \quad N_{\text{eq}} = -8.7, \quad N_{\text{rec}} = -7.0, \quad (59)$$

where the subscript ‘eq’ denotes matter-radiation equality.

Equation (58) can easily be solved numerically. However, using the approximation that $\rho_r \gg \rho_m \gg \rho_\Lambda$ for $N_* \leq N \leq N_{\text{eq}}$ and $\rho_m \gg \rho_r \gg \rho_\Lambda$ for $N_{\text{eq}} \leq N \leq N_{\text{rec}}$ we can also find an analytical solution that agrees very well with the exact numerical one. The result is

$$\tilde{S}(N) = \begin{cases} \tilde{S}^* + \frac{1}{3}\tilde{S}_{,N}^* (e^{3(N-N_*)} - 1) & \text{for } N_* \leq N \leq N_{\text{eq}}, \\ \tilde{S}^* - \frac{1}{3}\tilde{S}_{,N}^* + \frac{1}{4}\tilde{S}_{,N}^* e^{3(N_{\text{eq}}-N_*)} \left(\frac{1}{3} + e^{4(N-N_{\text{eq}})}\right) & \text{for } N_{\text{eq}} \leq N \leq N_{\text{rec}}, \end{cases} \quad (60)$$

where the superscript $*$ denotes evaluation at the beginning $N_* \sim -60$ of the radiation-dominated era. To determine if the time dependence of \tilde{S} leads to significant effects we must have an estimate for $\tilde{S}_{,N}^*$. Evaluating (54) and (50) at N_* (where $\rho_r \gg \rho_m \gg \rho_\Lambda$) for the situation under consideration and combining them we find

$$\tilde{S}_{,N}^* = -48 \frac{\Omega_\Lambda}{\Omega_m} e^{3N_*} \tilde{S}^*. \quad (61)$$

Here we have taken $\delta\rho_\Lambda^* = 0$, assuming Λ to be a pure cosmological constant. This means that

$$\tilde{S}(N_{\text{rec}}) = \tilde{S}^* - 12 \frac{\Omega_\Lambda}{\Omega_m} e^{4N_{\text{rec}}-N_{\text{eq}}} \tilde{S}^* = (1 - 10^{-7}) \tilde{S}^*. \quad (62)$$

Hence the effect of the fact that \tilde{S} is not constant in this system is completely negligible. This system is practically equivalent to one without a cosmological constant, which was of course to be expected as the energy density of the cosmological constant is so much smaller than that of matter and radiation before recombination. (As soon as one takes a third component with $\delta\rho_3^*/\rho_3^* \gg \tilde{S}^*$ or with $w_3 \neq -1$, it is no longer possible to express $\tilde{S}_{,N}^*$ in terms of \tilde{S}^* only. Even though the solution for $\tilde{S}(N)$ can still be calculated explicitly in these cases, it becomes more difficult to determine the relative importance of the time dependence of \tilde{S} .)

5. Summary and conclusions

In this paper, we investigated some consequences of multiple-field inflation for the cosmic microwave background radiation. Building on the theory of [10] we derived expressions for the amplitudes (25)–(28) and the spectral indices (32)–(35), (37) of the CMB power spectrum, all valid to next-to-leading order in slow roll. We also derived expressions valid to leading order for the derivatives of the spectral indices (A.1)–(A.4) in the appendix. All this in the context of a very general inflation theory with an arbitrary number of real scalar fields that may be the coordinates of a non-trivial field manifold (i.e. have non-minimal kinetic terms). There are four different versions of all these spectral quantities: three related to the scalar perturbations (adiabatic, (total) isocurvature and mixing between those two) and one related to the tensor perturbations.

These expressions were discussed in subsection 3.2. To summarize, multiple-field effects can be important for the scalar spectral quantities, not only for the isocurvature and mixing components (which are absent in the single-field case), but also for the adiabatic ones. In all multiple-field terms the slow-roll function $\tilde{\eta}^\perp$, which measures the size of the acceleration perpendicular to the field velocity, plays a key role; if it is negligible during the last 60 e-folds of inflation, multiple-field effects are unimportant. Unfortunately, to work out the expressions for the multiple-field terms explicitly, especially for the isocurvature perturbations, a careful analysis of the transition at the end of inflation as well as of the era of (p)reheating is in general required. However, as was shown in [10], there is a wide class of models where $\tilde{\eta}^\perp$ goes to zero at the end of inflation (while being non-negligible before that). Then the integral expression for the multiple-field contributions to the adiabatic perturbation can be worked out explicitly without knowing the details of the transition, while isocurvature perturbations are expected to be unimportant in this case (barring possible amplification mechanisms during preheating). Even in those models multiple-field effects in the adiabatic component can be of leading-order importance. The leading-order terms of the spectral indices are of first order in slow roll, while their derivatives are of second order. Hence multiple-field slow-roll inflation generically predicts a CMB power spectrum that is close to scale-invariant. The tensor spectral quantities do not depend on multiple-field effects.

From the expressions for the amplitudes and spectral indices we derived some important relations, which are valid very generally and do not depend on the details of what happens at the end of and after inflation. Most important is equation (38) (or its extension to next-to-leading order (39)), which gives the size of the multiple-field contribution to the adiabatic perturbation in terms of the (in principle) observable spectral quantities. In other words, this relation provides an observational test to determine if multiple-field effects play a significant role during the last 60 e-folds of inflation. It does require, however, a sufficiently accurate measurement of the tensor spectral quantities.

In the case of only one or two fields, we have the special situation that there is one

more observational quantity than there are inflationary parameters. This means that we can then derive a consistency relation between the various spectral quantities, which can in principle be checked observationally, allowing one to distinguish between the cases of one, two, or more fields, provided that the observations are sufficiently accurate. We derived that consistency relation in subsection 3.3. If the derivative of the spectral index can also be measured, there is an additional consistency relation, which was derived in the appendix. With all these relations one should keep in mind that, while a large effect is probably a clear proof of multiple-field effects being important, a small effect might also signify the presence of trans-Planckian or short-distance physics (see e.g. [52, 57]).

An important ingredient of the calculation of the isocurvature amplitude was the observation that, although there may be many individual isocurvature perturbations in a multi-component system, only the total entropy perturbation \tilde{S} enters into the equation of motion for the gravitational potential. With the assumptions that the various components behave as ideal fluids and have no interactions, the time derivative of \tilde{S} on super-horizon scales during radiation and matter domination was worked out in (54). It was found that, if there are only two different types of energy in the universe (i.e. two different equations of state, e.g. baryons and cold dark matter with $p = 0$ and photons and hot dark matter with $p = \frac{1}{3}\rho$), \tilde{S} is simply constant. Although this is no longer true if there is a third type of energy, for example a cosmological constant with $p = -\rho$, we showed that for the general three-component case the equation of motion for $\tilde{S}(t)$ only depends on the background energy densities and can be computed explicitly, independently of the gravitational potential. An explicit calculation of a three-component case with matter, radiation, and a pure cosmological constant (with realistic values of the parameters) showed that the time evolution of \tilde{S} before recombination can be completely neglected in that case.

To be more general we also derived, at the end of subsection 2.2, an expression for the isocurvature perturbation in the case that \tilde{S} is not constant. It turned out that the effects of this can be absorbed in the definition of the multiple-field term, so that the general form of the amplitudes and spectral indices is unchanged. This means that the observational relations in subsection 3.3 do not depend on whether \tilde{S} is constant or not.

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Appendix A. Expressions for $d\tilde{n}/d \ln k$

The recent WMAP results have made it interesting to look at the derivatives of the spectral indices with respect to k as well. In a straightforward calculation (using (7) for the derivatives of the slow-roll functions), analogous to the derivation of the spectral

indices themselves in subsection 3.1, we find to leading (i.e. second) order in slow roll:

$$\begin{aligned} \left[\frac{d\tilde{n}_{\text{ad}}}{d \ln k} \right]_{\text{l.o.}} &= 2 \frac{4\tilde{\epsilon}_{\mathcal{H}}^2 + 3\tilde{\epsilon}_{\mathcal{H}}\tilde{\eta}_{\mathcal{H}}^{\parallel} + 3(\tilde{\eta}_{\mathcal{H}}^{\parallel})^2 + 3(\tilde{\eta}_{\mathcal{H}}^{\perp})^2 - \tilde{\xi}_{\mathcal{H}}^{\parallel}}{1 + U_{Pe}^T U_{Pe}} \\ &\quad + 2 \frac{2e_2^T \left(-\tilde{\xi}_{2\mathcal{H}} + \tilde{\eta}_{\mathcal{H}}^{\perp} \left(\tilde{\epsilon}_{\mathcal{H}} + 3\tilde{\eta}_{\mathcal{H}}^{\parallel} + 3\delta_{\mathcal{H}} \right) \right) U_{Pe} + U_{Pe}^T \left(2\delta_{\mathcal{H}}^2 - \frac{\dot{\delta}_{\mathcal{H}}}{H_{\mathcal{H}}} \right) U_{Pe}}{1 + U_{Pe}^T U_{Pe}} \\ &\quad - 4 \left(\frac{2\tilde{\epsilon}_{\mathcal{H}} + \tilde{\eta}_{\mathcal{H}}^{\parallel} + 2\tilde{\eta}_{\mathcal{H}}^{\perp} e_2^T U_{Pe} + U_{Pe}^T \delta_{\mathcal{H}} U_{Pe}}{1 + U_{Pe}^T U_{Pe}} \right)^2, \end{aligned} \quad (\text{A.1})$$

$$\left[\frac{d\tilde{n}_{\text{iso}}}{d \ln k} \right]_{\text{l.o.}} = 2 \frac{V_e^T \left(2\delta_{\mathcal{H}}^2 - \frac{\dot{\delta}_{\mathcal{H}}}{H_{\mathcal{H}}} \right) V_e}{V_e^T V_e} - 4 \left(\frac{V_e^T \delta_{\mathcal{H}} V_e}{V_e^T V_e} \right)^2, \quad (\text{A.2})$$

$$\begin{aligned} \left[\frac{d\tilde{n}_{\text{mix}}}{d \ln k} \right]_{\text{l.o.}} &= 2 \frac{e_2^T \left(-\tilde{\xi}_{2\mathcal{H}} + \tilde{\eta}_{\mathcal{H}}^{\perp} \left(\tilde{\epsilon}_{\mathcal{H}} + 3\tilde{\eta}_{\mathcal{H}}^{\parallel} + 3\delta_{\mathcal{H}} \right) \right) V_e + U_{Pe}^T \left(2\delta_{\mathcal{H}}^2 - \frac{\dot{\delta}_{\mathcal{H}}}{H_{\mathcal{H}}} \right) V_e}{U_{Pe}^T V_e} \\ &\quad - 4 \left(\frac{\tilde{\eta}_{\mathcal{H}}^{\perp} e_2^T V_e + U_{Pe}^T \delta_{\mathcal{H}} V_e}{U_{Pe}^T V_e} \right)^2, \end{aligned} \quad (\text{A.3})$$

$$\left[\frac{d\tilde{n}_{\text{tens}}}{d \ln k} \right]_{\text{l.o.}} = -4\tilde{\epsilon}_{\mathcal{H}} \left(\tilde{\epsilon}_{\mathcal{H}} + \tilde{\eta}_{\mathcal{H}}^{\parallel} \right). \quad (\text{A.4})$$

In these equations the derivative $\dot{\delta}_{\mathcal{H}} \equiv d\delta_{\mathcal{H}}/dt_{\mathcal{H}}$ occurs. In specific models it can be calculated from the potential and the field metric using the definitions (11):

$$\frac{\dot{\delta}_{\mathcal{H}}}{H_{\mathcal{H}}} = 2\tilde{\epsilon}_{\mathcal{H}}\delta_{\mathcal{H}} + 2\tilde{\epsilon}_{\mathcal{H}}\tilde{\eta}_{\mathcal{H}}^{\parallel} (\mathbb{1} + 2e_1 e_1^T) - \frac{\dot{M}_{\mathcal{H}}^2}{3H_{\mathcal{H}}^3}, \quad (\text{A.5})$$

$$\dot{M}_{mn}^2 = (\mathcal{D}_t \mathbf{e}_m) \cdot \tilde{\mathbf{M}}^2 \mathbf{e}_n + \mathbf{e}_m \cdot (\mathcal{D}_t \tilde{\mathbf{M}}^2) \mathbf{e}_n + \mathbf{e}_m \cdot \tilde{\mathbf{M}}^2 (\mathcal{D}_t \mathbf{e}_n), \quad (\text{A.6})$$

$$(\mathcal{D}_t \tilde{\mathbf{M}}^2)_b^a = G^{ac} (\nabla_b \nabla_c \nabla_e V) \dot{\phi}^e - (\nabla_e R_{cdb}^a) \dot{\phi}^c \dot{\phi}^d \dot{\phi}^e - 2R_{cdb}^a (\mathcal{D}_t \dot{\phi}^c) \dot{\phi}^d. \quad (\text{A.7})$$

The derivatives of the basis vectors are given in equation (4).

If we also include the $d\tilde{n}_X/d \ln k$ as observational quantities, there are, in the single-field case, two of those and one additional inflationary parameter ($\tilde{\xi}_{\mathcal{H}}^{\parallel}$), so that there should be one more consistency relation (in addition to (41)). It can be written as

$$\frac{1}{\tilde{n}_{\text{tens}}} \frac{d\tilde{n}_{\text{tens}}}{d \ln k} = \tilde{n}_{\text{tens}} - \tilde{n}_{\text{ad}} \quad (\text{one field}). \quad (\text{A.8})$$

In the two-field case, there are two more observational quantities but also two more inflationary parameters ($\tilde{\xi}_{2\mathcal{H}}$ and $\dot{\delta}_{\mathcal{H}}$), so that we also have a single additional consistency relation:

$$\frac{1}{\tilde{n}_{\text{tens}}} \frac{d\tilde{n}_{\text{tens}}}{d \ln k} = \frac{\tilde{n}_{\text{tens}} - \tilde{n}_{\text{ad}} + r_{\text{mix}}^2 (2\tilde{n}_{\text{mix}} - \tilde{n}_{\text{tens}} - \tilde{n}_{\text{iso}})}{1 - r_{\text{mix}}^2} \quad (\text{two fields}), \quad (\text{A.9})$$

where r_{mix}^2 is defined in (43).

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